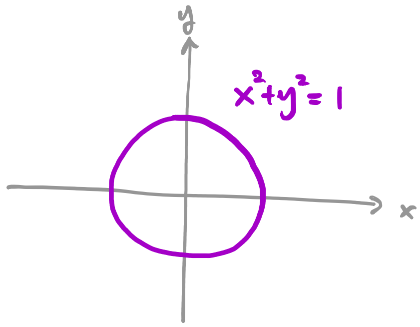
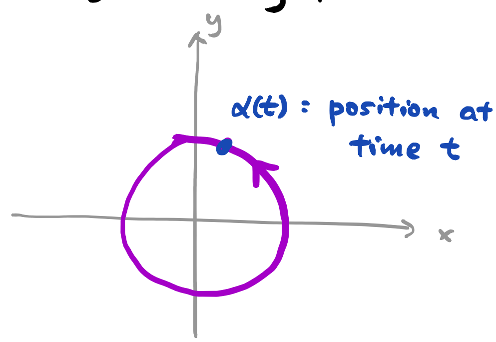
§ Curves and Arc Length (do Carmo § 1.2, 1.3)

Question: What is a "curve" (in $\mathbb{R}^3$)?

There are two different ways to think of a curve:

(I) as a geometric locus OR (II) as the path described by a moving particle





Remark: We are mostly interested in the geometric shape of a curve as in (I), but (II) is more useful since we can bring in the tools from "Calculus" to describe the geometric behaviour. It is like giving a "coordinate system" along a curve which allows calculations to be done.

Definition: A (smooth parametrized) curve is a C^∞ map

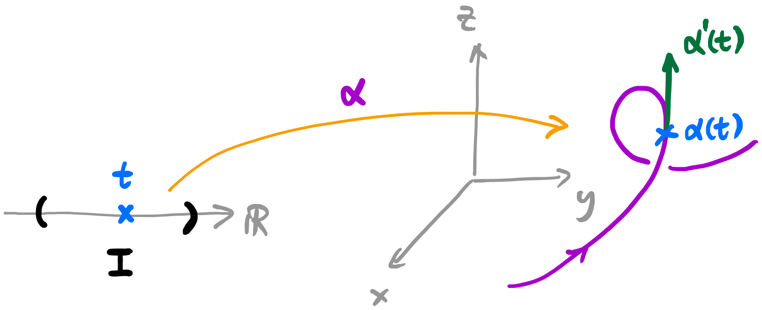
$$\alpha : I \longrightarrow \mathbb{R}^3$$

$$\alpha(t) = (x(t), y(t), z(t))$$

where $I \subset \mathbb{R}$ is a (connected) open interval, which is possibly unbounded.

tangent vector of α at $t \in I$: $\alpha'(t) = (x'(t), y'(t), z'(t))$

trace of α : $\text{image}(\alpha) = \alpha(I) \subset \mathbb{R}^3$.



a space curve

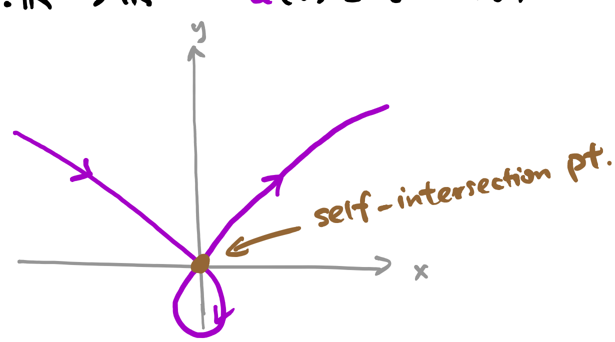
<u>Note</u>: We say that $\alpha : I \to \mathbb{R}^3$ is a plane curve if there exists a plane $P \subseteq \mathbb{R}^3$ s.t. $\alpha(I) \subseteq P$.

After a rigid motion (rotation + translation) in $\mathbb{R}^3$, we can assume that $P =$ xy-plane, i.e.

$$\alpha(t) = (x(t), y(t), 0) \quad , \text{ i.e. } \alpha : I \to \mathbb{R}^2 .$$

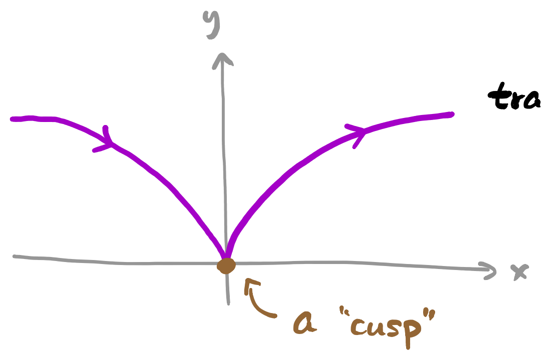
<u>Remarks</u>: 1) A curve may have self-intersections, i.e. α may not be 1-1.

<u>E.g.</u> $\alpha : \mathbb{R} \to \mathbb{R}^2 \qquad \alpha(t) = (t^3 - 4t, t^2 - 4)$



2) The trace of α may not be smooth.

E.g.: $\alpha : \mathbb{R} \to \mathbb{R}^2$, $\alpha(t) = (t^3, t^2)$



$\text{trace}(\alpha) = \{ y = x^{2/3} \}$

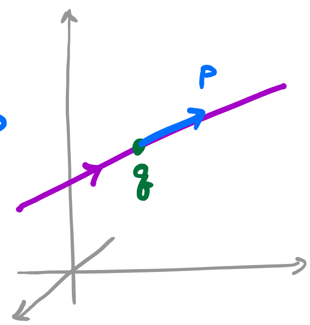
not smooth at $x = 0$!

There are three important curves which we will keep mentioning from time to time.

Example I: Straight lines

$\alpha : \mathbb{R} \to \mathbb{R}^3$, $\alpha(t) = q + t \cdot p$

is a line through q and parallel to p.

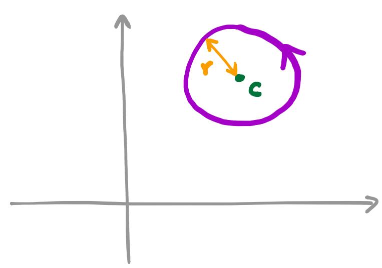


Example II: Circles

$\alpha : \mathbb{R} \to \mathbb{R}^2$

$\alpha(t) = c + r(\cos t, \sin t)$

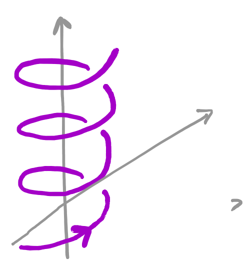
circle of radius $r > 0$ centered at c.



Example III: Helix

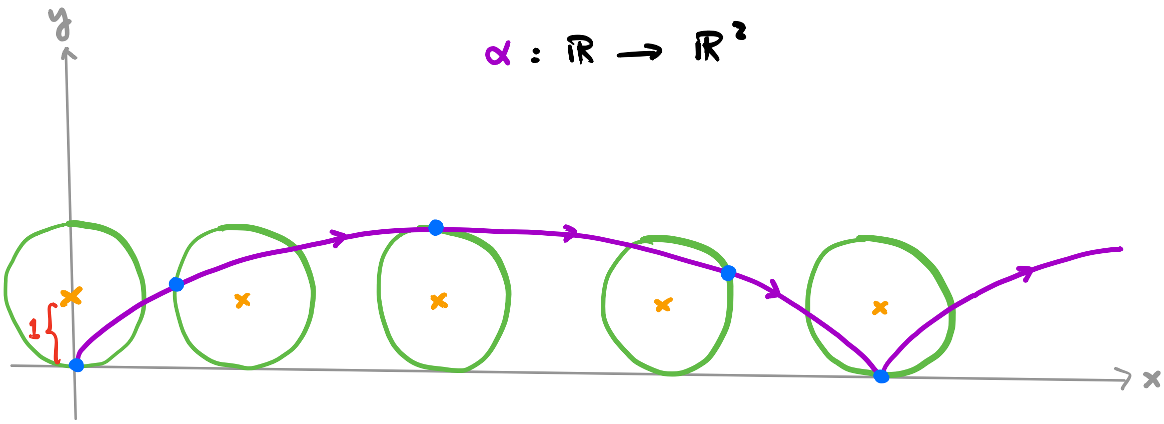
$\alpha : \mathbb{R} \to \mathbb{R}^3$

$\alpha(t) = (\cos t, \sin t, t)$



A more complicated example: Cycloid

$\alpha : \mathbb{R} \to \mathbb{R}^2$



$$\alpha(t) = \underbrace{(t, 1)}_{\text{moving center}} + \underbrace{(-\sin t, -\cos t)}_{\text{rolling on ball}} = (t - \sin t, 1 - \cos t)$$

Definition: Let $\alpha : I \to \mathbb{R}^3$ be a curve, and $[a,b] \subset I$.

The length of α from a to b is defined as

$$(*): \quad \boxed{L_a^b(\alpha) := \int_a^b |\alpha'(t)|\, dt}$$

Remark: The notion of length defined above is "geometrical" as it depends only on the geometric locus of the curve. More precisely, the length of a curve is invariant under 1 rigid motions and 2 reparametrization.

We now explain these two concepts.

Example: Arc length of helix from Example III

$$L_0^{2\pi}(\alpha) = \int_0^{2\pi} |\alpha'(t)|\, dt = \int_0^{2\pi} \sqrt{2}\, dt = 2\sqrt{2}\,\pi$$